E9 203 Compressed Sensing & Sparse SP

21 Apr. 2021.

Coherence

Defn. $A \in \mathbb{C}^{m\times N}$ with l_2 normalized cols $a_1, a_2, \dots, a_N$ s.t. $\|a_i\|_2 = 1$, $i \in [N]$. The coherence $\mu(A)$ is defined as

$$\mu \triangleq \mu(A) \triangleq \max_{1\le i\ne j\le N} |\langle a_i, a_j\rangle|$$

The l_1-coherence fn. (Babel fn.)

$$\mu_1(s) = \max_{i\in[N]} \max\left\{ \boxed{\sum_{j\in S} |\langle a_i, a_j\rangle|},\ S\subset[N],\ |S| = s,\ i\notin S \right\},$$

where $s \in [N-1]$.

$$\mu_1(s) = \max_{\substack{S: |S| = s \\ S\subset[N]}} \max_{j\notin S} \sum_{i\in S} |\langle a_i, a_j\rangle| .$$

Note:

1. For $1 \le s \le N-1$

$$\mu \le \mu_1(s) \le s\mu \qquad [HW]$$
$$\mu = \mu_1(1)$$

2. For $1 \le s, t \le N-1$, $s+t \le N-1$

$$\max\{\mu_1(s), \mu_1(t)\} \le \mu_1(s+t) \le \mu_1(s) + \mu_1(t).$$

3. Coherence & l_1-coherence are invariant to left mult. by a unitary matrix $\mathcal{U}$.
   $\because$ $\mathcal{U}a_1, \dots, \mathcal{U}a_N$ are l_2 normalized
   & $\langle \mathcal{U}a_i, \mathcal{U}a_j\rangle = \langle a_i, a_j\rangle$

4. $\mu \le 1$ [$\because$ of Cauchy-Schwarz ineq.]

5. $\mu = 0$ iff A has orthonormal cols.
   $\Rightarrow$ when $m < N$, $\mu > 0$.

6. Small $\mu \Rightarrow$ col submatrices of A are well-conditioned.

Thm. 5.3 $A \in \mathbb{C}^{m\times N}$, l_2 normalized cols, $s \in [N]$.
For all s-sparse $x \in \mathbb{C}^N$,

$$(1 - \mu_1(s-1))\|x\|_2^2 \le \|Ax\|_2^2 \le (1 + \mu_1(s-1))\|x\|_2^2 .$$

Equivalently, for each $S \subset [N]$ with $|S| \le s$, the EVals of $A_S^H A_S \in [1-\mu_1(s-1), 1+\mu_1(s-1)]$.

In particular, if $\mu_1(s-1) < 1$, then $A_S^H A_S$ is invertible.

Proof: For $S \subset [N]$, $|S| \le s$, $A_S^H A_S$ is PSD

$\Rightarrow$ Has an orthonormal basis of EVecs,
EVals are real, ≥ 0.
Let λ_{min}, λ_{max} : smallest & largest EVal.

For any $x \in \mathbb{C}^N$, $supp(x) = S$

$$\|Ax\|_2^2 = \langle A_S x_S, A_S x_S\rangle = \langle \underbrace{A_S^H A_S}_{\text{diag.} = 1} x_S, x_S\rangle$$

$\because$ cols of A have unit l_2 norm.

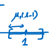
By Gersgorin's disk thm.,

$$\text{EVals}(A_S^H A_S) \in \bigcup_{i=1}^{|S|} \text{disk}_i$$

where disk_i is centered 1, with radius

$$r_i \triangleq \sum_{\substack{l\in S \\ l\ne i}} |(A_S^H A_S)_{i,l}|$$

$$= \boxed{\sum_{\substack{l\in S \\ l\ne i}} |\langle a_l, a_i\rangle|} \le \mu_1(s-1)\ \forall\, i\in S.$$



Since the EVals are real, they must lie in the interval $[1-\mu_1(s-1),\ 1+\mu_1(s-1)]$. ☒

Cor. 5.4 Given $A \in \mathbb{C}^{m\times N}$ l_2 norm. cols, integer $s \ge 1$, if $\mu_1(s) + \mu_1(s-1) < 1$, then, for each set $S \subset [N]$, $|S| \le 2s$, the matrix $A_S^H A_S$ is invertible and A_S is injective.
(Follows $\because$ if $\mu_1(s) + \mu_1(s-1) < 1$, then $\mu_1(2s-1) < 1$.)

Thm. 5.7

The coherence of $A \in \mathbb{C}^{m\times N}$ with l_2-normalized cols satisfies

$$\mu \ge \sqrt{\frac{N-m}{m(N-1)}} .$$

[The ineq. becomes an equality for a family of matrices called Grassmanian frames or equiangular tight frames (ETF).]

[For $N \gg m$, $\mu \gtrsim O\left(\frac{1}{\sqrt{m}}\right)$. "Quadratic bottleneck"]

Proof: Let $G = A^H A \in \mathbb{C}^{N\times N}$ (Gram matrix)

$$H = AA^H \in \mathbb{C}^{m\times m}$$
$$G_{ij} = \overline{\langle a_i, a_j\rangle} = \langle a_j, a_i\rangle,\ i,j \in [N].$$

Since the cols $a_1 \dots a_N$ are l_2 normalized,

$$tr(G) = \sum_{i=1}^{N} \|a_i\|_2^2 = N . \quad — ①$$

Define the inner product:

$$\langle U, V\rangle_F = tr(UV^H) = \sum_{i,j=1}^{n} u_{ij}\overline{v_{ij}}$$

$\langle \cdot, \cdot\rangle_F$ induces the norm $\|\cdot\|_F$ as

$$\|U\|_F = (tr(UU^H))^{\frac{1}{2}} .$$

Using the C-S inequality

$$tr\, H = \langle H, I_{m\times m}\rangle_F \le \|H\|_F \cdot \|I_{m\times m}\|_F$$
$$= \sqrt{tr(HH^H)}\,\sqrt{m} \quad — ②$$

Observe that

$$tr(HH^H) = tr(AA^HAA^H) = tr(A^HAA^HA) = tr(GG^H)$$
$$= \sum_{i,j=1}^{N} |\langle a_i, a_j\rangle|^2$$
$$= \sum_{i=1}^{N} \|a_i\|_2^2 + \sum^{N} |\langle a_i, a_j\rangle|^2$$

$$= \sum_{i=1} \underbrace{\qquad} \qquad \sum_{\substack{i,j=1 \\ i \neq j}}$$

$$= N + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} |\langle a_i, a_j \rangle|^2 \text{ --- } \underline{③}$$

Since $\mathrm{tr}(G) = \mathrm{tr}(H)$, combining ①, ②, ③

$$N^2 = (\mathrm{tr}(G))^2 = (\mathrm{tr}(H))^2 \leq \underline{\underline{\mathrm{tr}(HH^H)\, m}}$$

$$= m \left(N + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \underline{\underline{|\langle a_i, a_j \rangle|^2}} \right)$$

Since $|\langle a_i, a_j \rangle| \leq \underline{\mu} \ \forall\, i,j \in [N],\ i \neq j$

$$N^2 \leq m \left(N + N(N-1)\mu^2 \right)$$

$$\Rightarrow \mu^2 \geq \frac{N^2 - mN}{mN(N-1)} = \frac{N-m}{m(N-1)} . \qquad \boxtimes$$

Similar bound for ℓ_1-coherence:

$$\mu_1(s) \geq s \sqrt{\frac{N-m}{m(N-1)}} \quad \text{for } s < \sqrt{N-1} .$$

See text for the proof.

Defn. The <u>spark</u> of a matrix $A \in \mathbb{C}^{m \times N}$ is the smallest k s.t. A has a set of k linearly dependent cols.

$$\mathrm{Spark}(A) = \min_{\underline{\underline{x \neq 0}}} \|x\|_0 \quad \text{s.t. } \underline{\underline{Ax = 0}} .$$

NP hard to compute.

If all cols. of A are LI, $\mathrm{spark}(A) \triangleq \infty$.

$$\mathrm{Spark}(A) \geq 1 + \frac{1}{\mu(A)} .$$